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13. ABSTRACT *(Maximum 200 words)*
New nonlinear signal processing and modeling techniques were examined. The key issues which formed the focus of the project: were 1] the incorporation of symmetries into the modeling process. Such symmetires might be deduced from fundamental principles, or inferred from observations. The incorporation of such symmetries leads to simpler, more robust models, with fewer free parameters. 2] The design of coupling terms for synchronizing the model with driving signals from the system of interest. This is the first analytical result of its kind, and gives sufficient conditions for guaranteeing that the model will synchronize. 3] The development of a basic theory of symbolic time series analysis for Markov systems, 4] the use of symbolic methods to extract correlation timescales and to detect masked periodic signals.

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Improved Techniques for Modeling and Controlling Nonlinear Systems with Few Degrees of Freedom

Final Report for Grant No. AFOSR F49620-96-1-0116
from
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Executive Summary

Statement of objectives

The research carried out under this grant concerned the development of improved modeling and control techniques. The techniques are built upon observational data and are tailored to the problem at hand. The models used are global discrete time mappings, ordinary and partial differential equations.

One major effort of the proposed program was the exploitation of symmetries to constrain the fitted models. These models were shown to be more compact and could accurately model behaviors not explicitly shown in the data.

A second major effort was an examination of synchronization between identical dynamical systems. The focus has been on developing a method for designing coupling schemes that guarantee synchronous motion between the systems. The results are analytical and represent a method for determining the complete state of a nonlinear system from limited measurements.

A third major effort concerned the development of symbolic time series methods. This technique has great promise in high-noise situations, where it has been shown to be capable of robust parameter estimation for both low- and high-dimensional systems. During the period of this project, we concentrated on developing a theory of symbolic time series analysis for Markov systems, examined the effects on the symbolic data of variation of both the time delay for sampling and the partition, as well as initiating the investigation of detecting of changes of state.

Methods employed

The models used are expansions in polynomials, or have been derived from first principles. The basis set used for the expansion is constructed to be orthonormal on the measured data after embedding into an appropriate state space. The coefficients of the models are fit to the data by either a least squares, or an annealing procedure using the raw data or its coarse-grained symbolic form. In addition, the fitting procedure accounts for the size and complexity of the expansion models. This results in models that are optimal in the sense that they are the simplest models (within a given class of models) that are consistent with the data.

An important and novel aspect of the research program is the array of tests and constraints implemented to ensure that the models are corret and contain the right physics. In order to insure that the models are close approximations to the true equations of motion in the reconstructed phase space a series of *a priori* constraints and/or *a posteriori* tests are imposed. The *a priori* constraints involve determining from the data any symmetries the attractor may exhibit and restricting the class of models to those that respect the symmetries. The *a posteriori* test compares the coarse-grained symbol statistics generated by the model with that of the data.

Significance of the proposed activity

The proposed research will significantly enhance the ability to detect, model and control the dynamics of low-dimensional nonlinear systems using observed time series data. The synchronization of two nonlinear oscillators can be exploited as a means of nondestructive testing of devices, or as a real time monitor of dynamics, or as a mechanism for controlling dynamics. Symbolic time series methods are attractive because they are relatively robust to noise and involve only discrete calculations. Hence, they should be very fast in real-time applications. The approach used is comprehensive and will be implemented on experimentally obtained data from a diverse group of sources.

Contents

C. INTRODUCTORY COMMENTS:

The project covered by this report (AFOSR Grant No. F49620-96-1-011) was carried out during the time frame April 1, 1996-March 31, 1997. Part of this period overlapped the time frame of the earlier effort covered by Grant No. AFOSR F49620-95-1-0261 (final report dated Dec. 19, 1996) due to a no-cost extension granted on the earlier project.

In the interests of completeness of exposition we have included a brief summary of the earlier work as well, as the present final report builds directly on the earlier results. New results, developed under Grant No. F49620-96-1-011, are clearly indicated in the text and summarized in Section G.

D. RESEARCH OBJECTIVES

The goal of this research program is the development of new robust algorithms for diagnosing, modeling, and controlling nonlinear systems. The focus was on systems that have only a few degrees of freedom, but some of the techniques developed have wider applicability. (In this context the number of *degrees of freedom* is equal to the number of independent variables needed to accurately model the system which generated the data.)

Time-series data, in conjunction with whatever *a priori* information is known about the system, are used as input to the modeling process. These methods are robust to uncertainties in the models, and to the presence of noise [1–5]. These techniques have been successfully applied to data from chemical reactions, electronic circuits, and mechanical systems [1]. Other researchers are also beginning to apply these methods to experimental data [6–8].

E. BACKGROUND

Successful modeling relies on mathematical results from Mañé [9], Takens [10], Sauer, Yorke, and Casdagli [11], and others who have shown that it is often possible to reconstruct the full multidimensional dynamics of a nonlinear system from a *single* scalar time series, $s(n) = s(n\tau)$ for $n = 1, 2, \ldots$ (τ is the sampling interval associated with the measurements) [12]. The most common technique uses time delays to form vectors $\mathbf{y}(n)$

$$\mathbf{y}(n) = [s(n),\, s(n+T),\, \ldots, s(n+(d-1)T)],$$

where $\mathbf{y}(n) \in I\!R^d$. Time evolution, as given by $\mathbf{y}(n) \mapsto \mathbf{y}(n+1)$, is equivalent (diffeomorphic) to the true evolution of the system that produced the scalar time series. Fraser and Swinney [13] have developed a technique for determining T from data, while Kennel *et al.* [14] have developed a technique for determining d.

1. Low-dimensional modeling

The dynamics is modeled by *global* discrete-time maps

$$\mathbf{y}(n+1) = \mathbf{F}[\mathbf{y}(n)],$$

or *global* ordinary differential equations (ODE's)

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}).$$

When modeling the observed system via ODE's, time evolution of the data is modeled as a single implicit Adams integration time step

$$\mathbf{y}(n+1) = \mathbf{y}(n) + \tau \sum_{j=0}^{M} a_j^{(M)} \mathbf{F}[\mathbf{y}(n+1-j)].$$

The $a_j^{(M)}$'s are the Adams integration coefficients, and are known for all values of j and M.

For a generic time series one does not know the functional form of $\mathbf{F}$. Hence, the best that one can hope for is a series expansion in some set of basis functions, $\phi^{(\mathbf{I})}(\mathbf{z})$,

$$\mathbf{F}(\mathbf{z}) = \sum_{\mathbf{I}=0}^{\mathbf{Np}} \mathbf{p}^{(\mathbf{I})} \phi^{(\mathbf{I})}(\mathbf{z}).$$

The data approximates an invariant distribution on the attractor that defines the system under observation

$$\rho(\mathbf{z}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \delta[\mathbf{z} - \mathbf{y}(n)],$$

where N is the number of data vectors. The basis functions $\phi^{(\mathbf{I})}(\mathbf{z})$, are polynomials constructed (via Gram-Schmidt) to be orthonormal on the attractor defined by the data,

$$\begin{aligned} \left\langle \phi^{(\mathbf{I})} | \phi^{(\mathbf{J})} \right\rangle &= \int d\mathbf{z}\, \rho(\mathbf{z})\, \phi^{(\mathbf{I})}(\mathbf{z})\, \phi^{(\mathbf{J})}(\mathbf{z}) \\ &= \delta_{\mathbf{I}\,\mathbf{J}}. \end{aligned}$$

In this equation $\mathbf{I}$ and $\mathbf{J}$ are index vectors used to indicate the order of the polynomials.

It has been shown that this formulation of the modeling problem allows more accurate and robust determination of $\mathbf{F}$, for larger sampling intervals, than previous methods [1,7,8,15–17].

2. Symbolic time series analysis

In symbolic time-series analysis the state space of the system is partitioned into a finite number of cells and a symbol, s, is assigned to each cell. Such a symbolic approach is appealing because the symbol statistics are robust in the presence of noise and easy to estimate from observations. Our previous work [4] showed that the symbolic data are quite robust and that even at high noise levels (signal/noise$\approx \mathcal{O}(1)$) the effects of noise on the symbolic data can be tracked and effectively eliminated.

To briefly summarize the approach, we convert the observed analog signal stream, x_n, into a symbol sequence by passing it through a threshold function which takes $\{x_n\} = (x_1, x_2, \cdots, x_N) \to \{s_n\} = (s_1, s_2, \cdots, s_N)$ where $s_k \in (a, b)$. For example, if $x_n < x*$ then $s_n \equiv a$ and if $x_n > x*$ then $s_n \equiv b$. From the symbol sequence, $\{s_n\}$, construct the *symbol tree*

$$\begin{array}{cccccccc} & & p_a & & & & p_b & \\ & p_{aa} & & p_{ab} & & p_{ba} & & p_{bb} \\ p_{aaa} & p_{aab} & p_{aba} & p_{abb} & p_{baa} & p_{bab} & p_{bba} & p_{bba} \end{array}$$

etc.

Here p_{aab} is the probability of observing the sequence aab, etc. The symbol tree is a compact summary of (coarse grained) information about multiple-time-step correlations in the signal. The

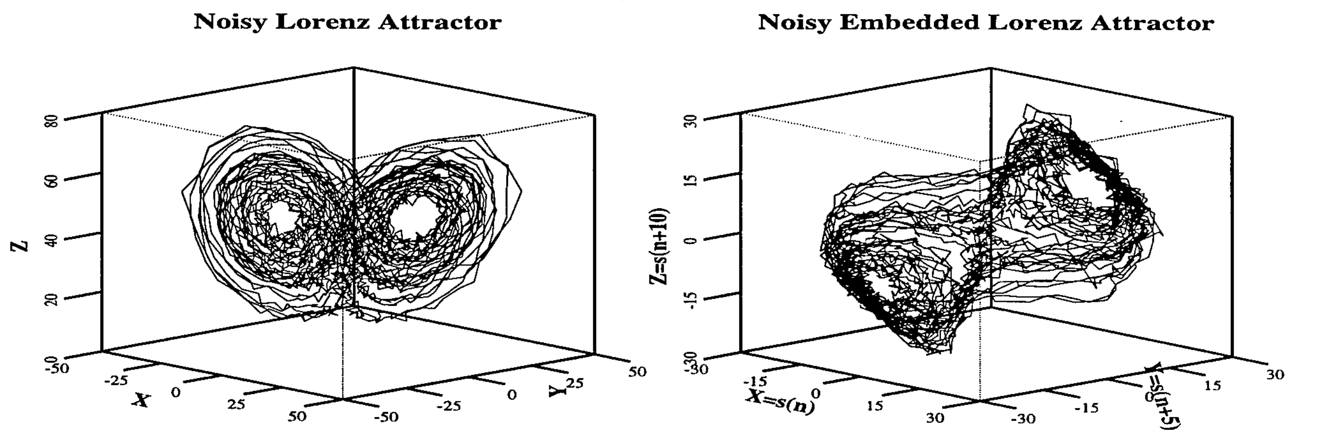

FIG. 1.

higher levels of the tree are determined by the lowest level measured, hence we focus only on the bottom level used (typically going to level 5 has proven to be sufficient for most purposes). In what follows, we denote the vector formed by the probabilities at the lowest level by $\mathbf{P}$, e.g. at level 5: $\mathbf{P} \equiv (p_{aaaaa}, \ldots, p_{bbbbb}) \in \mathbb{R}^{32}$. Not all of these quantities are independent, as will be discussed in Section F 5.

F. RESULTS

1. Using symmetries to constrain expansion models

Symmetries represent an important class of physical properties that can be used to constrain models of dynamical systems. Using a symmetry to constrain a model means restricting the functional form of the model to one that, *a priori*, contains the symmetry. They can be determined from the data and/or from first principles. We have found that if a symmetry is present in the data or the dynamics then an equivariante model constructed from the data accurately models the dynamics [2].

To illustrate how symmetries constrain a model, suppose the attractor, A, is invariant under the action of the symmetry $\mathbf{S}$ ($\mathbf{S} \circ A = A$). This condition results in the following equivariance condition on $\mathbf{F}$

$$\mathbf{F}(\mathbf{S} \circ \mathbf{y}) = \mathbf{S} \circ \mathbf{F}(\mathbf{y}). \tag{1}$$

Clearly, Eq. (1) places restrictions on the functional form of $\mathbf{F}$.

The Lorenz system is a dynamical system where we used symmetry to restrict the functional form of $\mathbf{F}$. Figure 1(a) shows a noisy representations of the true Lorenz attractor. Despite the noise, Fig. 1(a) indicates that if $\mathbf{S}$ is a rotation about the $\hat{z}$ axis by π radians then $\mathbf{S} \circ A = A$. The origin of this symmetry is the $x \to -x$ and $y \to -y$ invariance of the Lorenz equations. For this system Eq. (1) implies that all terms proportional to $x^n y^m$ must vanish for even values of $n + m$. Thus, on the basis of equivariance half of the terms in the model can be eliminated. Of course, if the data $\mathbf{y} = [x, y, z]$ is noise free then we expect the coefficients of these terms to vanish. However, *experimental data is never noise free.* Therefore, it is best to, *a priori*, eliminate terms from the model.

Figure 1(b) shows a Lorenz attractor that has been reconstructed by embedding a scalar time series given by the x coordinate of the data shown in Fig. 1(a). The time series is symmetric about zero which leads to the inversion symmetry ($\mathbf{S} = -1$) evident in Fig. 1(b). Using an ensemble

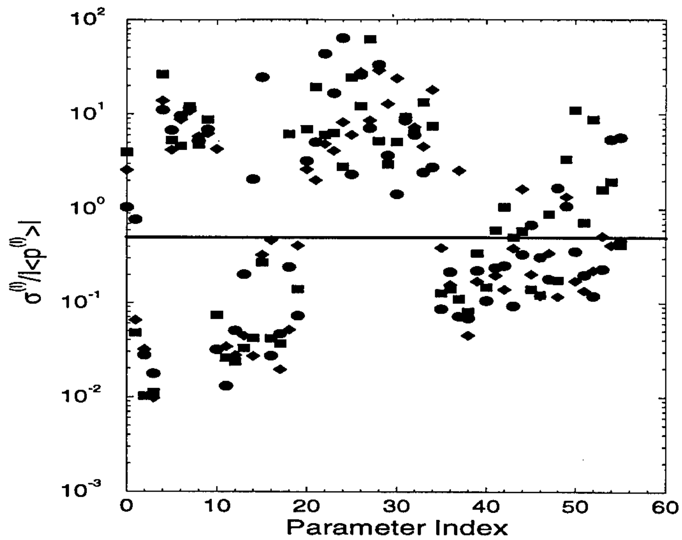

FIG. 2. Graphs of parameter uncertainty versus the index associated with the parameter.

of data sets we constructed models without the restriction of Eq. (1). The results are shown in Fig. 2, where $\left\langle \mathbf{p}^{(\mathrm{I})} \right\rangle$ is the mean value of the parameter $\mathbf{p}^{(\mathrm{I})}$ and $\boldsymbol{\sigma}^{(\mathrm{I})}$ is the standard deviation of the parameter about its mean value. The circles squares and diamonds indicate the first, second, and third components of $\mathbf{p}^{(\mathrm{I})}$, respectively. The solid line represents $\boldsymbol{\sigma}^{(\mathrm{I})}/|\left\langle \mathbf{p}^{(\mathrm{I})} \right\rangle| = 2$.

For symbols above this line $|\left\langle \mathbf{p}^{(\mathrm{I})} \right\rangle| \pm \boldsymbol{\sigma}^{(\mathrm{I})}$ straddles zero. Hence, the value of $\mathbf{p}^{(\mathrm{I})}$ obtained by the fitting procedure is statistically indistinguishable from zero and one conjectures that their values are dominated by noise, finite sample size, and round off effects. This is supported by noticing that *all* of the symbols for index values 0, 4–9, and 20–34 are above the line. These indices correspond to terms in $\mathbf{F}$ whose coefficients, $\mathbf{p}^{(\mathrm{I})}$, should vanish because their basis function, $\phi^{(\mathrm{I})}$, is not equivariant under $\mathbf{S} = -\mathbf{1}$.

For comparison we also estimated the uncertainty of the $\mathbf{p}^{(\mathrm{I})}$'s using standard statistical methods [2]. These uncertainties, $\boldsymbol{\mu}^{(\mathrm{I})}$, were found to be good estimates of $\boldsymbol{\sigma}^{(\mathrm{I})}$. Thus, in some cases, it is possible to detect the presence of a symmetry from a single data set by examining $\boldsymbol{\mu}^{(\mathrm{I})}/|\left\langle \mathbf{p}^{(\mathrm{I})} \right\rangle|$ [2].

We also examined data from the electronic circuit shown in Fig. 3(a). It is possible to determine from first principles that, for certain values of α, the dynamics lives on one of the two disjoint attractors shown in Fig. 3(b) [18]. For this value of α a clear plane of symmetry can be seen between the attractors and a single experimentally measured time series would correspond to one or the other of these attractors, but not both. As α increases the two attractors merge into a single attractor via a symmetry increasing bifurcation (also called a crisis [19]).

Our work examined data from before and after the bifurcation. When examining data taken before the bifurcation we found that equivariant models constructed using data from one of the attractors accurately mimics the dynamics on *both* attractors [2]. This is not the case for previous modeling techniques, which require data from both attractors and two models [2]. Furthermore, the equivariant model contains fewer coefficients than either of the nonequivariant models. When modeling data taken after the bifurcation we found that we were able to detect the presence of the symmetry using the methods discussed above for the Lorenz example.

G. ACCOMPLISHMENTS AND NEW FINDINGS

The primary accomplishments of the grant period (4/1/96 - 3/31/97) are:

1. Low-dimensional modeling
   - Exploitation of symmetries to constrain the modeling process. Modeling algorithms were developed which detect and impose symmetries on the functional form of the model. This has been shown to improve the stability of the modeling process, as well as ensuring that important properties of the dynamical system are represented by the model *even if they are not directly present in the data set* [2].
   - A rigorous sufficient criteria has been derived which guarantees linearly stable synchronization between dynamical systems when they are coupled in a drive–response manner. This result is an analytic method for solving the observer problem for nonlinear plants [3].
2. Symbolic time series analysis
   - Development of a basic theory for the symbolic time series analysis of Markov systems.
   - Demonstration that the symbolic approach can be used to develop robust esimates of correlation times and detect weak periodic signals masked by noise or chaotic signals.

H. PERSONNEL SUPPORTED

Senior Personnel
E. R. Tracy (1 month).
Reggie Brown (3 months).

Post-doctoral fellow
Xian-zhu Tang (12 months).

I. PUBLICATIONS

1. R. Brown and N. F. Rulkov, "Synchronization of chaotic systems: transverse stability of trajectories in invariant manifolds", to appear in *Chaos.*
2. R. Brown, V. In and E. R. Tracy, "Parameter uncertainties in models of equivariant dynamical systems", *Physica D* **102**, ??? (1997).
3. X.-Z. Tang, E. R. Tracy and R. Brown, "Symbol statistics and spatio-temporal systems", *Physica D* **102**, 253 (1997).
4. R. Brown, "Using Models to diagnose, test, and control chaotic systems", to appear in the proceedings of the Third Experimental Chaos Conference.
5. G. Gouesbet, L. Le Sceller, C. Letellier and R. Brown, "Reconstruction of a set of equations from scalar time series", to appear in the proceedings of the Eleventh Annual Florida Workshop on Nonlinear Astronomy.
6. E. R. Tracy, "Toward a theory of symbolic time series analysis", in preparation.
7. X.-Z. Tang and E. R. Tracy, "Data compression and information retrieval via symbolization", in preparation.

J. INTERACTIONS/TRANSITIONS

1. Presentations at meetings

1. Poster presentation (contributed) at Sherwood (Fusion Theory) Meeting, Philadelphia, PA (March, 1996), E. Tracy, X.-Z. Tang, R. Brown & S. Burton.
2. Poster presentation (contributed) at American Physical Society, Division of Plasma Physics Meeting, Denver, CO (November, 1996), E. Tracy.
3. Seminars at AlliedSignal Engines, Phoenix, AZ, March, 1996, R. Brown and E. Tracy.

1. Wm & Mary/AlliedSignal:
Performer:
Professors E. R. Tracy and Reggie Brown, Dr. X.-Z. Tang & Ms. Sharon Burton.
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Contact:
Dr. R. Burne
Research Manager
(410)964-4159.
Anticipated result: Using the symbol statistics to detect transitions in complex systems, *e.g.* noise-driven turbulent flows.
Application: Early detection of rotating stall in turbines.
Ms. Burton was supported by AlliedSignal MTC as an intern during the summer of 1996.

2. Oak Ridge National Lab/Ford Motor Company:
Work performed as part of a Cooperative Research and Development Agreement (CRADA), number ORNL-95-0337 titled "Engine Control Improvement Through Application of Chaotic Time Series Analysis".
Performer:
Dr. C. Stuart Daw
Oak Ridge National Laboratory
P.O. Box 2009
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Dr. John Hoard
(313)594-1316.

Anticipated result: Using symbol statistics to do parameter fitting for an internal combustion engine model [6]. (Aspects of the work are subject to a patent disclosure.)
Application: Improved feedback control for internal combustion engines to reduce NO_x and hydrocarbon emission and increase fuel efficiency.

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